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PRECISE ITERATION FORMULAE OF THE MASLOV-TYPE INDEX THEORY FOR SYMPLECTIC PATHS

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Abstract

In this paper, using homotopy components of symplectic matrices, and basic properties of the Maslov-type index theory, we establish precise iteration formulae of the Maslov-type index theory for any path in the symplectic group starting from the identity.

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1. Main results.

We consider linear Hamiltonian systems

$$\dot{x} = JB(t)x, \qquad x \in \mathbf{R}^{2n}, \tag{1.1}$$

with $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$, where $S_\tau = \mathbf{R}/(\tau\mathbf{Z})$ for $\tau > 0$, $\mathcal{L}(\mathbf{R}^{2n})$ denotes the set of $2n \times 2n$ real matrices, and $\mathcal{L}_s(\mathbf{R}^{2n})$ denotes its subset of symmetric ones. It is well known that the fundamental solution γ_B of (1.1) is a path in the symplectic group

$$\mathrm{Sp}(2n) = \{M \in \mathcal{L}(\mathbf{R}^{2n}) \,|\, M^T JM = J\}$$

with $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, i.e. $\gamma_B \in \mathcal{P}_\tau(2n)$ with $\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \,|\, \gamma(0) = I\}$.

In the study of periodic solutions of Hamiltonian systems, an index theory for such symplectic paths was introduced by C. Conley and E. Zehnder in [CZ] for non-degenerate elements in $\mathcal{P}_\tau(2n)$ with $n \geq 2$, by E. Zehnder and the author in [LZ] for non-degenerate elements in $\mathcal{P}_\tau(2)$, by the author in [Lo1] and C. Viterbo in [Vi2] independently for degenerate symplectic paths which are fundamental solutions of Hamiltonian systems, and by the author in [Lo8] for any symplectic paths together with an axiom characterization of this index theory. We call this index theory the Maslov-type index theory and denote it by $(i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbf{Z} \times \{0, \ldots, 2n\}$.

In many problems related to nonlinear Hamiltonian or Lagrangian systems, it is necessary to study iterations of periodic solutions. In order to solve such problems, one way is to study the Maslov-type indices of iterations of fundamental solutions of corresponding linearized systems. For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, we define the iteration of γ by

$$\tilde{\gamma}(t) = \gamma(t - j\tau)\gamma(\tau)^j, \qquad \forall j\tau \leq t \leq (j+1)\tau,\ j = 0, 1, 2, \ldots, \tag{1.2}$$

$$\gamma^m = \tilde{\gamma}|_{[0, m\tau]}, \qquad \forall m \in \mathbf{N}. \tag{1.3}$$

Here $\mathbf{N}$ denotes the set of all natural numbers. Correspondingly we obtain the sequence of index pairs:

$$(i_{m\tau}(\gamma^m), \nu_{m\tau}(\gamma^m)) \in \mathbf{Z} \times \{0, 1, \ldots, 2n\}, \quad \forall m \in \mathbf{N}. \tag{1.4}$$

Note that if $\gamma : [0, +\infty) \to \mathrm{Sp}(2n)$ is the fundamental solution of the system (1.1) for some $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$, then by the uniqueness of the initial value problem of (1.1), the path γ coincides with its iteration $\tilde{\gamma}$ defined by (1.2) completely on $[0, \infty)$.

The iteration theory of Morse indices of second order Hermitian systems was first established by R. Bott in his pioneering work [Bo] of 1956. Bott's idea was used by I. Ekeland in [Ek1] to [Ek3] of 1980's to his index theory for convex Hamiltonian systems, and by C. Viterbo in [Vi1] to certain dual index for non-degenerate star-shaped Hamiltonian systems in 1989. We also refer to [CD] and [BTZ] for related works.

In the full generality of (1.4), studies on the Maslov-type index theory for iterations of any symplectic paths started from [DL], where D. Dong and the author established iteration inequalities of this index theory. In [Lo5], the author further studied such iteration inequalities. In [Lo9] the author extended this index theory to a new family of index functions parametrized by elements on the unit circle $\mathbf{U}$ in the complex plane $\mathbf{C}$, and established the Bott-type iteration formulae of the Maslov-type index theory in terms of these index functions. Based on these results, various sharp iteration inequalities were established by C. Liu and the author in [LL1] and [LL2]. These results have been applied to the study of various problems of Hamiltonian systems (cf. [DL], [Lo5], [Lo7], etc.) Nevertheless, the precise iteration equalities of the Maslov-type index theory for any symplectic path in $\mathcal{P}_\tau(2n)$ is only known when the path is non-degenerate or hyperbolic, which has been established in [DL], [Lo5], and [Lo7].

Our aim in this paper is to establish the precise iteration formulae (the following Theorem 1.3) of the Maslov-type index theory for iterations of any symplectic path by a rather simple and elementary homotopy method. At the same time, our result gives a precise representation of the Maslov-type mean index of a symplectic path defined in [Lo9] in terms of topological invariants of the end matrix of this symplectic path in its homotopy component in the symplectic group. This result will be used in our forthcoming papers on nonlinear problems.

To describe our main results, for any two symplectic matrices of square block form:

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i\times 2i}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j\times 2j},$$

the $\diamond$-product of M_1 and M_2 is defined by the $2(i+j)\times 2(i+j)$ matrix:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k-fold $\diamond$-product $M\diamond\cdots\diamond M$. Note that the $\diamond$-multiplication is associative, and the $\diamond$-product of any two symplectic matrices is symplectic. For $a\in\mathbf{R}\backslash\{0\}$ denote by $D(a) = \mathrm{diag}(a, 1/a)$. The following so called basic normal forms are studied in [Lo9]:

$$N_1(\lambda, c) = \begin{pmatrix} \lambda & c \\ 0 & \lambda \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad N_2(\omega, B) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix},$$

where $\lambda = \pm 1$, $c\in\mathbf{R}$, $\omega = e^{\theta\sqrt{-1}}\in\mathbf{U}\backslash\mathbf{R}$ with $\theta\in\mathbf{R}$, $b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ is a 2×2 real matrix with $b_2 - b_3 \neq 0$. In [Lo9], the normal form $N_2(\omega, b)$ is called **nontrivial** if $(b_2 - b_3)\sin\theta < 0$, or **trivial** if $(b_2 - b_3)\sin\theta > 0$.

Definition 1.1. (Definition 1.1 of [Lo9]) For any $M\in\mathrm{Sp}(2n)$, define the **homotopy set** of M in $\mathrm{Sp}(2n)$ by

$$\begin{aligned}\Omega(M) = \{N\in\mathrm{Sp}(2n)\,|\,\sigma(N)\cap\mathbf{U} = \sigma(M)\cap\mathbf{U},\ \mathit{and}\\ \dim_{\mathbf{C}}\ker_{\mathbf{C}}(N-\lambda I) = \dim_{\mathbf{C}}\ker_{\mathbf{C}}(M-\lambda I),\ \forall\,\lambda\in\sigma(M)\cap\mathbf{U}\}.\end{aligned}$$

We denote by $\Omega^0(M)$ the path connected component of $\Omega(M)$ which contains M, and call it the **homotopy component** of M in $\mathrm{Sp}(2n)$.

For any $M\in\mathrm{Sp}(2n)$, define $[M] = \{N\in\mathrm{Sp}(2n)\,|\,N = P^{-1}MP \text{ for some } P\in\mathrm{Sp}(2n)\}$. Then $[M]\subset\Omega^0(M)$. Let $[r] = \max\{m\in\mathbf{Z}\,|\,m\le r\}$ for every $r\in\mathbf{R}$. We have

Theorem 1.2. (Theorem 7.8 of [Lo9]) *For any $M \in \mathrm{Sp}(2n)$, there is an $f \in C([0,1],\Omega^0(M))$ such that $f(0)=M$ and*

$$\begin{aligned} f(1) =& N_1(1,1)^{\diamond p_-}\diamond I_{2p_0}\diamond N_1(1,-1)^{\diamond p_+}\diamond N_1(-1,1)^{\diamond q_-}\diamond(-I_{2q_0})\diamond N_1(-1,-1)^{\diamond q_+} \\ &\diamond R(\theta_1)\diamond\cdots\diamond R(\theta_r)\diamond N_2(\omega_1,u_1)\diamond\cdots\diamond N_2(\omega_{r_*},u_{r_*}) \\ &\diamond N_2(\lambda_1,v_1)\diamond\cdots\diamond N_2(\lambda_{r_0},v_{r_0})\diamond M_0, \end{aligned} \tag{1.5}$$

where $p_-, p_0, p_+, q_-, q_0, q_+, r, r_*$, and r_0 are nonnegative integers; $\omega_j = e^{\sqrt{-1}\alpha_j}$, $\lambda_j = e^{\sqrt{-1}\beta_j}$; θ_j, α_j, $\beta_j \in (0,\pi)\cup(\pi,2\pi)$; $N_2(\omega_j,u_j)$'s are nontrivial and $N_2(\lambda_j,v_j)$'s are trivial basic normal forms; $\sigma(M_0)\cap\mathbf{U}=\emptyset$. The integers p_-, p_0, p_+, q_-, q_0, q_+, r, r_*, r_0, and the real numbers θ_j for $1\le j\le r$, α_h for $1\le h\le r_*$, β_k for $1\le k\le r_0$, are uniquely determined by M.

We denote by

$$I(m,\theta) \equiv -\left[\left[\frac{m\theta}{2\pi}\right]-\frac{m\theta}{2\pi}\right] \in \{0,\ 1\}, \quad \forall m\in\mathbf{N}, \theta\in\mathbf{R}. \tag{1.6}$$

Note that $I(m,\theta)=0$ if $m\theta=0 \bmod 2\pi$, and $I(m,\theta)=1$ otherwise. The following is the main result in this paper.

Theorem 1.3. *For $\tau>0$, let $\gamma\in\mathcal{P}_\tau(2n)$. In Theorem 1.2 we let $M=\gamma(\tau)$ and use notations there. Then for any $m\in\mathbf{N}$ there hold*

$$\begin{aligned} i_{m\tau}(\gamma^m) =& m(i_\tau(\gamma)+p_-+p_0-r)+2\sum_{j=1}^{r}[\frac{m\theta_j}{2\pi}] \\ &-p_--p_0-\frac{1+(-1)^m}{2}(q_0+q_+) \\ &+2\sum_{j=1}^{r}I(m,\theta_j)-r+2(\sum_{j=1}^{r_*}I(m,\alpha_j)-r_*), \end{aligned} \tag{1.7}$$

$$\nu_{m\tau}(\gamma^m) = \nu_\tau(\gamma)+\frac{1+(-1)^m}{2}(q_-+2q_0+q_+)+2\varphi(m,\gamma(\tau)), \tag{1.8}$$

where we denote by

$$\varphi(m,\gamma(\tau)) = (r-\sum_{j=1}^{r}I(m,\theta_j))+(r_*-\sum_{j=1}^{r_*}I(m,\alpha_j))+(r_0-\sum_{j=1}^{r_0}I(m,\beta_j)). \tag{1.9}$$

By the studies in [Lo9], [LA], and [Vi2], the Morse index theory for calculus of variations in [Bo], the Ekeland index theory in [Ek3], and the index theory for certain star-shaped Hamiltonian systems of [Vi1] are special cases of the Maslov-type index theory. Thus our formulae generalize their corresponding results.

In the following section 2, the most basic case for $\mathcal{P}_\tau(2)$ is studied. In the section 3, we study the case of truly hyperbolic and hyperbolic paths, and then elliptic paths in $\mathrm{Sp}(4)$. Based on these results and Theorem 1.2, we obtain Theorem 1.3. Finally in the section 4, we recover the iteration inequalities proved in [LL1] and [LL2] via Theorem 1.3.

2. Paths in $\mathrm{Sp}(2)$

We define

$$\mathrm{Sp}(2n)^{\pm} = \{M \in \mathcal{L}(\mathbf{R}^{2n}) \,|\, \pm(-1)^{n-1}\det(M-I) < 0\},$$
$$\mathrm{Sp}(2n)^{*} = Sp(2n)^{+} \cup \mathrm{Sp}(2n)^{-}, \qquad \mathrm{Sp}(2n)^{0} = \mathrm{Sp}(2n) \backslash \mathrm{Sp}(2n)^{*}.$$

For $\lambda = \pm 1$ and $\omega \in \mathbf{U}$, we define

$$\mathrm{Sp}(2)^{0}_{\lambda,\pm} = \{M \in \mathrm{Sp}(2) \,|\, P^{-1}MP = N_1(\lambda, \mp b) \text{ for some } P \in \mathrm{Sp}(2) \text{ and } b > 0\},$$
$$\mathcal{M}_{\omega}(2n) = \{M \in \mathrm{Sp}(2n) \,|\, \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I) = 1\}.$$

In this section, we study the Maslov-type indices for iterations of any path in $\mathrm{Sp}(2)$. We use notations introduced in the section 1. The main tool in our proof is the following concept of the homotopy.

Definition 2.1. (Definition 2 of [Lo1], Definition 1.2 of [Lo8]) Given two paths γ_0 and $\gamma_1 \in \mathcal{P}_{\tau}(2n)$, if there is a map $\delta \in C([0,1] \times [0,\tau], \mathrm{Sp}(2n))$ such that $\delta(0,\cdot) = \gamma_0(\cdot)$, $\delta(1,\cdot) = \gamma_1(\cdot)$, $\delta(s,0) = I$, and $\nu_{\tau}(\delta(s,\cdot))$ is constant for $0 \le s \le 1$, then γ_0 and γ_1 are **homotopic on** $[0,\tau]$ **along** $\delta(\cdot,\tau)$ and we write $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ along $\delta(\cdot,\tau)$. This homotopy possesses **fixed end points** if $\delta(s,\tau) = \gamma_0(\tau)$ for all $s \in [0,1]$.

The following lemmas are frequently used in our study on iterations.

Lemma 2.2. *For $\theta \in \mathbf{R}$, define $\phi_{\tau,\theta} : [0,+\infty) \to \mathrm{Sp}(2)$ by*

$$\phi_{\theta,\tau}(t) = R(\frac{t}{\tau}\theta), \qquad \forall\, t \in [0,1].$$

When $\tau = 1$, we write simply ϕ_{θ}. Then there holds

$$i_{\tau}(\phi_{\theta,\tau}) = -2[-\frac{\theta}{2\pi}] + 1 = \begin{cases} 2[\frac{\theta}{2\pi}] + 1 & \text{if } \theta \neq 0 \bmod 2\pi, \\ 2[\frac{\theta}{2\pi}] - 1 & \text{if } \theta = 0 \bmod 2\pi. \end{cases}$$

Proof. This follows from the $\mathbf{R}^3$-cylindrical coordinate representation of $\mathrm{Sp}(2)$ and a direct computation. ∎

Lemma 2.3. *Let α and $\beta \in \mathcal{P}_{\tau}(2n)$. Suppose*

$$\alpha(\tau) \in \Omega^0(\beta(\tau)), \quad i_{\tau}(\alpha) = i_{\tau}(\beta). \tag{2.1}$$

Then for any $m \in \mathbf{N}$ there hold

$$i_{m\tau}(\alpha^m) = i_{m\tau}(\beta^m), \quad \nu_{m\tau}(\alpha^m) = \nu_{m\tau}(\beta^m). \tag{2.2}$$

Proof. Set $\tau = 1$. By (2.1), there is a path $f : [0,1] \to \Omega^0(\beta(1))$ such that $f(0) = \alpha(1)$ and $f(1) = \beta(1)$. Thus by the inverse homotopy theorem (Theorem 6.4 of [Lo8]) and (2.1), there holds $\alpha \sim \beta$ along f via a homotopy map $\delta : [0,1]^2 \to \mathrm{Sp}(2n)$. Since $\delta(1,t)^m = f(t)^m \in \Omega^0(\alpha(1)^m)$ for any $t \in [0,1]$, extending δ to $[0,1] \times [0,m]$ yields a homotopy from α^m to β^m along f^m. Thus by the homotopy theorem (1^o of Theorem 1.4 of [Lo8]) we obtain (2.2). ∎

Lemma 2.4. *Let $\gamma \in \mathcal{P}_\tau(2n)$ such that $\gamma(\tau) = M_1 \diamond M_2$ with $M_j \in \mathrm{Sp}(2n_j)$ and $n_1 + n_2 = n$. Suppose paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 1$ and 2 satisfy $\gamma_j(\tau) = M_j$ and $i_\tau(\gamma) = i_\tau(\gamma_1) + i_\tau(\gamma_2)$. Then for any $m \in \mathbf{N}$ there hold*

$$i_{m\tau}(\gamma^m) = i_{m\tau}(\gamma_1^m) + i_{m\tau}(\gamma_2^m), \tag{2.3}$$

$$\nu_{m\tau}(\gamma^m) = \nu_{m\tau}(\gamma_1^m) + \nu_{m\tau}(\gamma_2^m). \tag{2.4}$$

Proof. Set $\tau = 1$. By the inverse homotopy theorem (Theorem 6.4 of [Lo8]), there holds $\gamma \sim (\gamma_1 \diamond \gamma_2)$ with fixed points via a homotopy map $\delta : [0,1]^2 \to \mathrm{Sp}(2n)$. Extending this δ to $[0,1]\times[0,m]$ yields a homotopy from γ^m to $(\gamma_1 \diamond \gamma_2)^m = \gamma_1^m \diamond \gamma_2^m$. By the homotopy invariant and symplectic additivity of the Maslov-type index theory (1^o and 2^o of Theorem 1.4 of [Lo8]), we then obtain (2.3) and (2.4). ∎

We need following notions on symplectic matrices and paths.

Definition 2.5. A matrix $M \in \mathrm{Sp}(2n)$ is **truly hyperbolic** if $\sigma(M) \cap \mathbf{U} = \emptyset$, **hyperbolic** if two eigenvalues of M are 1 and all the other eigenvalues of M are not on $\mathbf{U}$, **elliptic** if $\sigma(M) \subset \mathbf{U}$, **strongly elliptic** if $\sigma(M) \subset \mathbf{U}\backslash\{1,-1\}$, or **parabolic** if $\sigma(M) = \{-1\}$. We denote by $\mathrm{Sp}^{th}(2n)$, $\mathrm{Sp}^{h}(2n)$, $\mathrm{Sp}^{e}(2n)$, $\mathrm{Sp}^{se}(2n)$, and $\mathrm{Sp}^{pa}(2n)$ the set of all truly hyperbolic, hyperbolic, elliptic, strongly elliptic and parabolic symplectic matrices in $\mathrm{Sp}(2n)$ respectively. A path $\gamma \in \mathcal{P}_\tau(2n)$ is defined to be truly hyperbolic, hyperbolic, elliptic, strongly elliptic, or parabolic respectively, if $\gamma(\tau)$ belongs to the corresponding subset of $\mathrm{Sp}(2n)$.

According to the end matrix $\gamma(\tau)$, we study the iteration of corresponding path $\gamma \in \mathcal{P}_\tau(2)$ in four cases.

Case 1. *The degenerate case, $\gamma \in \mathcal{P}_\tau(2)$ with $\sigma(\gamma(\tau)) = \{1\}$.*

In this case we have the following result.

Theorem 2.6. *For $\gamma \in \mathcal{P}_\tau(2)$ with $\sigma(\gamma(\tau)) = \{1\}$, one of the following three cases must happen:*

1° *If $\gamma(\tau) \in \mathrm{Sp}(2)^0_{1,-}$, then there holds*

$$i_{m\tau}(\tilde{\gamma}) = m(i_\tau(\gamma) + 1) - 1, \qquad \nu_{m\tau}(\tilde{\gamma}) = 1, \qquad \forall m \in \mathbf{N}. \tag{2.5}$$

2° *If $\gamma(\tau) = I$, then there holds*

$$i_{m\tau}(\tilde{\gamma}) = m(i_\tau(\gamma) + 1) - 1, \qquad \nu_{m\tau}(\tilde{\gamma}) = 2, \qquad \forall m \in \mathbf{N}. \tag{2.6}$$

3° *If $\gamma(\tau) \in \mathrm{Sp}(2)^0_{1,+}$, then there holds*

$$i_{m\tau}(\tilde{\gamma}) = m i_\tau(\gamma), \qquad \nu_{m\tau}(\tilde{\gamma}) = 1, \qquad \forall m \in \mathbf{N}. \tag{2.7}$$

Proof. We only prove 1^o. The other two cases are similar and are left to the readers. Without loss of generality, set $\tau = 1$. Fix $\gamma \in \mathcal{P}^0_1(2)$. Define

$$\gamma_s(t) = \gamma(t) R(s\rho(t)\theta_0), \qquad \forall t \in [0,1], s \in [-1,1]. \tag{2.8}$$

where $\rho(t) = 3t^2 - 2t^3$ for $t \in [0,1]$, and $\theta_0 > 0$ is so small such that there holds $\gamma_s(1) \in \mathrm{Sp}(2)^*$ for all $s \in [-1,1]\backslash\{0\}$. Then by our study in [Lo1] and [Lo2], there hold

$$i_1(\gamma_s) = i_1(\gamma_{-s}) + 1 = i_1(\gamma) + 1, \quad \forall s \in (0,1], \tag{2.9}$$

$$\nu_1(\gamma_s) = 0, \quad \forall s \in [-1,1]\backslash\{0\}. \tag{2.10}$$

In the case of 1°, since $\gamma(1) \in \mathrm{Sp}(2)^0_-$, there holds $\gamma_{-1}(1) \in \mathrm{Sp}(2)^-$. Thus $i_1(\gamma)$ must be odd, and by the normal form theorem of [LD] there exists $P \in \mathrm{Sp}(2)$ such that $P^{-1}\gamma(1)P = N_1(1,1)$.

Therefore there hold $\gamma(1)^m \in \mathrm{Sp}(2)^0_-$ and $\nu_m(\gamma) = 1$ for all $m \in \mathbf{N}$.

Choose a smooth path $h : [0,1] \to \mathrm{Sp}(2)^0$ such that $h(0) = I$, $h(1) = \gamma(1)$, and $h(s) \in \mathrm{Sp}(2)^0_{1,-}$ for all $s \in (0,1]$. For $k = i_1(\gamma) + 1$ and this path h we define a new path $\delta_{k,h} : [0, +\infty) \to \mathrm{Sp}(2)$ by

$$\delta_{k,h}(t) = h * \phi_{k\pi,1}(t), \quad \forall t \in [0,1], \tag{2.11}$$

Using Lemma 2.3 and the definition of the Maslov-type index for the case of $\gamma(1) \in \mathrm{Sp}(2)^0_{1,-}$, we obtain

$$\gamma(0) = I = \delta_{k,h}(0), \qquad \gamma(1) = \delta_{k,h}(1),$$
$$i_1(\delta_{k,h}) = 2[\frac{i_1(\gamma)+1)}{2}] - 1 = i_1(\gamma).$$

Thus by Lemma 2.3, this implies

$$i_m(\gamma^m) = i_m(\delta^m_{k,h}) = 2[m(i_1+1)/2] - 1 = m(i_1+1) - 1, \qquad \forall m \in \mathbf{N}.$$

Thus (2.5) holds. ∎

Case 2. *The parabolic case, $\gamma \in \mathcal{P}_\tau(2)$ with $\sigma(\gamma(\tau)) = \{-1\}$.*

In this case we have the following result.

Theorem 2.7. *For $\gamma \in \mathcal{P}_\tau(2)$ with $\sigma(\gamma(\tau)) = \{-1\}$, one of the following three cases must happen:*

1° *If $\gamma(\tau) \in \mathrm{Sp}(2)^0_{-1,+}$, then there holds*

$$i_{m\tau}(\tilde\gamma) = m i_\tau(\gamma) - \frac{1+(-1)^m}{2}, \qquad \nu_{m\tau}(\tilde\gamma) = \frac{1+(-1)^m}{2}, \quad \forall m \in \mathbf{N}. \tag{2.12}$$

2° *If $\gamma(\tau) = -I_2$, then there holds*

$$i_{m\tau}(\tilde\gamma) = m i_\tau(\gamma) - \frac{1+(-1)^m}{2}, \qquad \nu_{m\tau}(\tilde\gamma) = 1+(-1)^m, \quad \forall m \in \mathbf{N}. \tag{2.13}$$

3° *If $\gamma(\tau) \in \mathrm{Sp}(2)^0_{-1,-}$, then there holds*

$$i_{m\tau}(\tilde\gamma) = m i_\tau(\gamma), \qquad \nu_{m\tau}(\tilde\gamma) = \frac{1+(-1)^m}{2}, \quad \forall m \in \mathbf{N}. \tag{2.14}$$

Proof. Note that there hold

$$N_1(-1,\pm 1) \in \mathrm{Sp}(2)^0_{-1,\mp} \quad \text{and} \quad N_1(-1,b)^m = N_1((-1)^m, (-1)^{m-1}mb), \quad \forall m \in \mathbf{N}, b \in \mathbf{R}.$$

Since the proof of Theorem 2.7 is similar to that of Theorem 2.6, it is left to readers. ∎

Case 3. *$\gamma \in \mathcal{P}_1(2)$ is truely hyperbolic.*

In this case we have the following result.

Theorem 2.8. *Suppose $\gamma \in \mathcal{P}_\tau(2)$ with $\gamma(\tau) \in \mathrm{Sp}^{th}(2)$. Then there holds*

$$i_{m\tau}(\tilde\gamma) = m i_\tau(\gamma), \qquad \nu_{m\tau}(\tilde\gamma) = 0, \qquad \forall\, m \in \mathbf{N}. \tag{2.15}$$

Proof. Set $\tau = 1$. Suppose $\sigma(\gamma(1)) = \{\lambda, \lambda^{-1}\}$ with $\lambda \in \mathbf{R}\backslash\{0\}$. We carry out the proof in two steps for the cases of $\lambda > 0$. The other case of $\lambda < 0$ is similar and left to the readers.

In this case, $\gamma(1) \in \mathrm{Sp}(2)^+$. Thus $i_1(\gamma)$ must be even, and by the normal form theorem in [HL], there exists $P \in \mathrm{Sp}(2)$ such that $P^{-1}\gamma(1)P = D(\lambda)$. Therefore there holds

$$\sigma(\gamma(m)) = \sigma(\gamma(1)^m) = \{\lambda^m, \lambda^{-m}\}, \qquad \forall\, m \in \mathbf{N}. \tag{2.16}$$

Then we obtain $\gamma(1)^m \in \mathrm{Sp}(2)^+$ and $\nu_m(\gamma) = 0$ for all $m \in \mathbf{N}$. Choose a smooth path $h : [0,1] \to \mathrm{Sp}(2)$ such that $h(0) = I$, $h(1) = \gamma(1)$, and $h(s) \in \mathrm{Sp}(2)^+$ for all $s \in (0,1]$. For $k = i_1$ and this path h we define a new path $\delta_{k,h} : [0, +\infty) \to \mathrm{Sp}(2)$ by (2.11). We obtain

$$\gamma(0) = I = \delta_{k,h}(0), \qquad \gamma(1) = \delta_{k,h}(1),$$
$$i_1(\delta_{k,h}) = (2[i_1(\gamma)/2] - 1) + 1 = i_1(\gamma).$$

Then by Lemma 2.3 this implies

$$i_m(\gamma) = i_m(\delta_{k,h}) = (2[mi_1(\gamma)/2] - 1) + 1 = mi_1(\gamma), \qquad \forall m \in \mathbf{N}.$$

Thus (2.15) holds. ∎

Case 4. *$\gamma \in \mathcal{P}_\tau(2)$ is strongly elliptic.*

In this case we have the following result.

Theorem 2.8. *Suppose $\gamma \in \mathcal{P}_\tau(2)$ satisfies $\sigma(\gamma(\tau)) = \{\omega, \omega^{-1}\}$ with $\omega = e^{\sqrt{-1}\theta}$ and $\theta \in (0,\pi) \cup (\pi, 2\pi)$. By the normal form theorem in [LD], there exists $P \in \mathrm{Sp}(2)$ such that*

$$P^{-1}\gamma(\tau)P = R(\hat{\theta}) \quad \text{with } \hat{\theta} = \theta \text{ or } 2\pi - \theta. \tag{2.17}$$

Then for any $m \in \mathbf{N}$, there hold:

$$i_{m\tau}(\tilde{\gamma}) = m(i_\tau(\gamma) - 1) + 2\left[\frac{m\hat{\theta}}{2\pi}\right] + 2I(m,\hat{\theta}) - 1, \tag{2.18}$$

$$\nu_{m\tau}(\tilde{\gamma}) = 2 - 2I(m,\hat{\theta}), \tag{2.19}$$

where $I(m,\hat{\theta})$ is defined in (1.6).

Proof. Set $\tau = 1$. In this case, $\gamma(1) \in \mathrm{Sp}(2)^-$. Thus $i_1(\gamma)$ must be odd. By (2.17) there holds $\sigma(\tilde{\gamma}(m)) = \sigma(\gamma(1)^m) = \{\omega^m, \omega^{-m}\}$ for all $m \in \mathbf{N}$. Define a path $\beta : [0, +\infty) \to \mathrm{Sp}(2)$ by

$$\beta(t) = R(t(i_1 - 1)\pi + t\hat{\theta}), \qquad \forall\, t \geq 0. \tag{2.20}$$

We obtain $\gamma(0) = I_2 = \beta(0)$, and

$$i_1(\beta) = 2[\frac{(i_1(\gamma) - 1)\pi + \hat{\theta}}{2\pi}] + 1 = i_1(\gamma).$$

Note that $\tilde{\beta} = \beta$. By (2.12) there is a path $h : [0,1] \to \Omega^0(\gamma(1))$ connecting $h(0) = \beta(1)$ to $h(1) = \gamma(1)$. Thus by Lemma 2.3, it suffices to prove (2.18) and (2.19) for β.

If $m\hat{\theta} \neq 0 \bmod 2\pi$, (2.20) yields $\beta(m) = \beta(1)^m \notin \mathrm{Sp}(2)^0$. Thus we obtain $\nu_m(\beta) = 0$ and

$$i_m(\beta) = 2[\frac{m(i_1(\beta)-1)\pi + m\hat{\theta}}{2\pi}] + 1 = m(i_1(\beta)-1) + 2[\frac{m\hat{\theta}}{2\pi}] + 1, \qquad \forall m \in \mathbf{N}.$$

If $m\hat{\theta} = 0 \bmod 2\pi$, (2.20) yields $\beta(m) = I_2$. We then obtain

$$\nu_m(\beta) = 2, \quad i_m(\beta) = 2[\frac{m((i_1(\beta)-1)\pi + \hat{\theta}}{2\pi}] - 1 = m(i_1(\beta)-1) + 2[\frac{m\hat{\theta}}{2\pi}] - 1, \qquad \forall m \in \mathbf{N}.$$

Thus by the definition (1.6) of $I(m, \hat{\theta})$, (2.18) and (2.19) hold for β. ∎

3. Truly hyperbolic, hyperbolic paths in $\mathrm{Sp}(2n)$ and elliptic paths in $\mathrm{Sp}(4)$

In this section, for $\tau > 0$ we establish the iteration formulae of the Maslov-type index theory for truly hyperbolic and hyperbolic paths.

Theorem 3.1. *For any $\gamma \in \mathcal{P}_\tau^{th}(2n)$, there hold*

$$i_{m\tau}(\tilde{\gamma}) = m i_\tau(\gamma), \qquad \nu_{m\tau}(\tilde{\gamma}) = 0, \qquad \forall m \in \mathbf{N}. \tag{3.1}$$

Proof. Set $\tau = 1$. Fix $\gamma \in \mathcal{P}_\tau^{th}(2n)$. By the proof of Lemma 2.3 of [LA], there is a path $f \in C([0,1], \mathrm{Sp}^{th}(2n))$ such that $f(0) = \gamma(\tau)$ and

$$f(1) = M_n^+, \quad i_1(\gamma) \in 2\mathbf{Z}, \qquad \text{if} \quad \alpha(\gamma(\tau)) = 0, \tag{3.2}$$
$$f(1) = M_n^-, \quad i_1(\gamma) \in 2\mathbf{Z}+1, \qquad \text{if} \quad \alpha(\gamma(\tau)) = 1. \tag{3.3}$$

where $M_n^+ = D(2)^{\diamond n}$, $M_n^- = D(-2) \diamond D(2)^{\diamond(n-1)}$, and $\alpha(\gamma(\tau))$ is the mod 2 number of the total multiplicity of eigenvalues of $\gamma(\tau)$ strictly less than -1, which is defined in [LA] and is called the **hyperbolic index** of $\gamma(\tau)$.

Since $\gamma(1)^m \in \mathrm{Sp}^{th}(2n)$, we obtain the second equality in (3.1).

Define $\beta_\pm(t) = D(\pm(1+t))$ for $0 \le t \le 1$. Define

$$\psi = (\beta_+ * \phi_{1,k\pi}) \diamond \beta_+^{\diamond(n-1)}, \quad \text{if} \quad \alpha(\gamma(\tau)) = 0, \tag{3.4}$$
$$\psi = (\beta_- * \phi_{1,k\pi}) \diamond \beta_+^{\diamond(n-2)}, \quad \text{if} \quad \alpha(\gamma(\tau)) = 1, \tag{3.5}$$

where $k = i_\tau(\gamma)$. Then by (3.2), (3.3), and Lemma 1.1 we obtain

$$i_{m\tau}(\gamma^m) = i_{m\tau}(\psi^m), \qquad \forall m \in \mathbf{N}. \tag{3.6}$$

Together with Theorem 1.4 of [Lo8], Lemmas 2.2 to 2.4, and Theorem 2.8, we obtain the first equality in (3.1). ∎

Fix $\gamma \in \mathcal{P}_\tau^h(2n)$. By Lemma 2.2 of [LA], there is a path $f \in C([0,1], \mathrm{Sp}^h(2n) \cap \Omega^0(\gamma(\tau)))$ when $n \ge 2$ such that $f(0) = \gamma(\tau)$ and

$$f(1) = N_1(1, b(\gamma(\tau)) \diamond M_{n-1}^+, \qquad \text{if} \quad \alpha(\gamma(\tau)) = 0, \tag{3.7}$$
$$f(1) = N_1(1, b(\gamma(\tau)) \diamond M_{n-1}^-, \qquad \text{if} \quad \alpha(\gamma(\tau)) = 1, \tag{3.8}$$

where $b(\gamma(\tau))=1$, 0, or -1 is uniquely determined by $\gamma(\tau)$. Note that there hold

$$i_\tau(\gamma)\in 2\mathbf{Z},\quad \text{if } b(\gamma(\tau))<0, \alpha(\gamma(\tau))=0, \text{ or } b(\gamma(\tau))\geq 0, \alpha(\gamma(\tau))=1, \tag{3.9}$$

$$i_\tau(\gamma)\in 2\mathbf{Z}+1, \text{ if } b(\gamma(\tau))\geq 0, \alpha(\gamma(\tau))=0, \text{ or } b(\gamma(\tau))<0, \alpha(\gamma(\tau))=1, \tag{3.10}$$

Theorem 3.2. *For any $\gamma\in\mathcal{P}_\tau^h(2n)$ and any $m\in\mathbf{N}$, there hold*

$$b(\gamma(\tau)^m)=b(\gamma(\tau)),\quad \nu_{m\tau}(\gamma^m)=2-|b(\gamma(\tau))|, \tag{3.11}$$

$$i_{m\tau}(\gamma^m)=m(i_\tau(\gamma)+1)-1,\quad \text{if}\ \ b(\gamma(\tau))\in\{0,\,1\}, \tag{3.12}$$

$$i_{m\tau}(\gamma^m)=mi_\tau(\gamma),\quad \text{if}\ \ b(\gamma(\tau))=-1. \tag{3.13}$$

Proof. Set $\tau=1$. When $n=1$, the conclusions follow from Theorem 2.6. When $n\geq 2$, (3.11) follows from the fact $f(1)^m=P(N_1(1,b(\gamma(1))\diamond M)P^{-1}$ for some $P\in\mathrm{Sp}(2n)$, $M\in\mathrm{Sp}(2(n-1))^*$, and all $m\in\mathbf{N}$. Let

$$g(t)=N_1(1,tb(\gamma(1))),\qquad \forall t\in[0,1].$$

and

$$\psi=\begin{cases} g\diamond(\beta^+ * \phi_{k\pi})\diamond\beta_+^{\diamond(n-2)}, & \text{if}\quad \alpha(\gamma(1))=0,\\ g\diamond(\beta_- * \phi_{k\pi})\diamond\beta_+^{\diamond(n-2)}, & \text{if}\quad \alpha(\gamma(1))=1,\end{cases} \tag{3.14}$$

where $\beta_\pm$ are defined in the proof of Theorem 3.1, $k=i_1(\gamma)+1$ if $b(\gamma(1))=0$ or 1, $k=i_1(\gamma)$ if $b(\gamma(1))=-1$. Thus by (3.9) and (3.10), the path ψ is well defined.

Then by Lemma 2.3, we also get (3.6). Together with Theorem 1.4 of [Lo8] and Theorems 2.6 and 2.8, we obtain (3.12) and (3.13). ∎

Based on Theorem 3.1, we further consider the following case in addition to the four basic cases on $\mathcal{P}_\tau(2)$ studied in the section 2.

Case 5. $\gamma\in\mathcal{P}_\tau(4)$ *with* $\gamma(\tau)=N_2(\omega,b)\in\mathcal{M}_\omega(4)$ *for* $\omega\in\mathbf{U}\backslash\mathbf{R}$.

Using notations introduced in Lemma 7.10 of [Lo9], let $\omega=\cos\theta+\sqrt{-1}\sin\theta$ with $\theta\in(0,\pi)\cup(\pi,2\pi)$ and $b=\begin{pmatrix} b_1 & b_2\\ b_3 & b_4\end{pmatrix}\in\mathcal{L}(\mathbf{R}^2)$. We have the following two theorems according to $N_2(\omega,b)$ being nontrivial or trivial.

Theorem 3.3. *Suppose $\gamma\in\mathcal{P}_\tau(4)$ satisfies $\gamma(\tau)=N_2(\omega,b)\in\mathcal{M}_\omega(4)$ with $\omega=e^{\sqrt{-1}\theta}$ and $\theta\in(0,\pi)\cup(\pi,2\pi)$. Suppose $N_2(\omega,b)$ is nontrivial, i.e. $(b_2-b_3)\sin\theta<0$. Then for any $m\in\mathbf{N}$ there hold*

$$i_{m\tau}(\gamma^m)=mi_\tau(\gamma)+2I(m,\theta)-2,\quad \nu_{m\tau}(\gamma^m)=2-2I(m,\theta). \tag{3.15}$$

where $I(m,\theta)\in\{0,1\}$ is defined by (1.6).

Proof. Without loss of generality, let $\tau=1$. By 4^o of Lemma 7.10 in [Lo9], there is a short enough perturbation path $f:[0,1]\to\mathrm{Sp}(4)$ such that $f(0)=N_2(\omega,b)$ and $f(t)$ possesses two distinct pairs of eigenvalues on $\mathbf{U}\backslash\mathbf{R}$ sufficiently close to ω for all $t\in(0,1]$. This then implies

$$i_1(\gamma)=i_1(f*\gamma)\in 2\mathbf{Z},\qquad \nu_1(\gamma)=0. \tag{3.16}$$

To continue our study, we fix an $m\in\mathbf{N}\backslash\{1\}$.

Since $N_2(\omega,b)$ is nontrivial, it can be connected to $N_2(\omega,R(\theta))$ by a path $\eta:[0,1]\to\Omega^0(N_2(\omega,b))$. Let $\beta=\eta*\gamma$. Then we obtain

$$i_k(\gamma^k)=i_k(\beta^k),\quad \nu_k(\gamma^k)=\nu_k(\beta^k),\quad \text{for}\ \ k=1,m.$$

Note that Lemma 2.3 has been applied when $k=m$. Thus it suffices to prove (3.12) for the path β. Note that there holds

$$\beta(1)^m=N_2(\omega,R(\theta))^m=\begin{pmatrix} R(m\theta) & mR(m\theta)\\ 0 & R(m\theta)\end{pmatrix},\quad \forall m\in\mathbf{N}. \tag{3.17}$$

Thus we obtain

$$\begin{aligned}\tilde{\beta}(m)=\beta(1)^m\in\mathrm{Sp}(2n)^0\quad &\text{if and only if}\quad m\theta=0 \bmod 2\pi,\\ &\text{if and only if}\quad \nu_m(\beta^m)=2.\end{aligned} \tag{3.18}$$

By (1.6), this implies the second equality of (3.15).

For small $\epsilon>0$ we define

$$\xi(t)=N_2(\omega,R(\theta))R(t\epsilon)^{\diamond 2},\qquad \forall t\in[-1,1],$$

and let

$$\xi_+(t)=\xi(t)\quad\text{and}\quad \xi_-(t)=\xi(-t)\quad \forall t\in[0,1]. \tag{3.19}$$

Then when ϵ is sufficiently small, by Theorems 1.4 and 6.6 of [Lo8] we obtain

$$i_1(\beta)=i_1(\xi_-*\beta)=i_1(\xi_+*\beta), \tag{3.20}$$

$$i_m(\beta^m)=i_m(\xi_-*(\beta^m))=i_m((\xi_-*\beta)^m). \tag{3.21}$$

By the computations in [Lo6] and from (4.10) to (4.13) and Lemma 7.10 in [Lo9], we obtain

$$\det(\xi(t)-\lambda I_4)=\lambda^4-4A_\xi(t)\lambda^3+B_\xi(t)\lambda^2-4A_\xi(t)\lambda+1. \tag{3.22}$$

As shown in [Lo9], the sign of the function

$$g(\sin(t\epsilon))\equiv 4A_\xi^2(t)+2-B_\xi(t),\qquad \forall t\in[-1,1], \tag{3.23}$$

determines the situation of eigenvalues of $\xi(t)$. Let $s=\sin\theta$ and $s_1=\sin(t\epsilon)$. Then for sufficiently small $\epsilon>0$ and $t\in[-1,1]$, by [Lo9] we obtain

$$g(0)=0,\qquad \frac{\partial g}{\partial s_1}(0)=s(b_2-b_3).$$

Therefore By the condition $s(b_2-b_3)<0$ and Lemma 7.9 of [Lo9], $\xi_+(t)$ possesses four eigenvalues outside $\mathbf{U}$ and $\xi_-(t)$ possesses four eigenvalues on $\mathbf{U}$ when $t\in(0,1]$. Specially, $\xi_+*\beta$ is a truly hyperbolic path in $\mathrm{Sp}(4)$.

Note that by the definition (3.8) of perturbation paths $(\beta^m)_{\pm 1}$ of β^m in [Lo8], we have

$$(\xi_+*\beta)^m\sim(\beta^m)_1,\qquad (\xi_-*\beta)^m\sim(\beta^m)_{-1}. \tag{3.24}$$

By Theorem 3.2 of [Lo8], we then obtain

$$i_m((\xi_- * \beta)^m) = i_m((\xi_+ * \beta)^m) - \nu_m(\beta^m). \tag{3.25}$$

Applying Theorem 3.1 to the hyperbolic path $\xi_+ * \beta$, by our above discussion we obtain

$$\begin{aligned} i_m(\beta^m) &= i_m((\xi_- * \beta)^m) \\ &= i_m((\xi_+ * \beta)^m) - \nu_m(\beta^m) \\ &= mi_1(\xi_+ * \beta) - \nu_m(\beta^m) \\ &= mi_1(\beta) - \nu_m(\beta^m). \end{aligned} \tag{3.26}$$

By the second equality of (3.15), we then obtain the first one of (3.15) for the path β. ∎

Theorem 3.4. *Suppose $\gamma \in \mathcal{P}_\tau(4)$ satisfies $\gamma(\tau) = N_2(\omega, b) \in \mathcal{M}_\omega(4)$ with $\omega = e^{\sqrt{-1}\theta}$ and $\theta \in (0, \pi) \cup (\pi, 2\pi)$. Suppose $N_2(\omega, b)$ is trivial, i.e. $(b_2 - b_3)\sin\theta > 0$. Then there hold*

$$i_{m\tau}(\gamma^m) = mi_\tau(\gamma), \quad \nu_{m\tau}(\gamma^m) = 2 - 2I(m, \theta). \tag{3.27}$$

Proof. Set $\tau = 1$. Since $N_2(\omega, b)$ is trivial, it can be connected to $N_2(\omega, -R(\theta))$ by a path $\eta : [0, 1] \to \Omega^0(N_2(\omega, b))$. Let $\beta = \eta * \gamma$. Then instead of (3.17) we obtain

$$\beta(1)^m = N_2(\omega, -R(\theta))^m = \begin{pmatrix} R(m\theta) & -mR(m\theta) \\ 1 & R(m\theta) \end{pmatrix}, \quad \forall m \in \mathbf{N}. \tag{3.28}$$

Thus we obtain

$$\begin{aligned} \tilde{\beta}(m) = \beta(1)^m \in \mathrm{Sp}(2n)^0 \quad &\text{if and only if} \quad m\theta = 0 \bmod 2\pi, \\ &\text{if and only if} \quad \nu_m(\beta^m) = 2. \end{aligned} \tag{3.29}$$

Thus the second equality in (3.27) holds. Using notations defined in the proof of Theorem 3.3, since $(b_2 - b_3)\sin\theta > 0$, the path $\xi_- * \beta$ is hyperbolic. Thus by (3.28), (3.29), and Theorem 3.1, we obtain

$$i_m(\beta^m) = i_m((\xi_- * \beta)^m) = mi_1(\xi_- * \beta) = mi_1(\beta).$$

This proves the first equality in (3.27). ∎

Based on our studies in the sections 2 and 3, for any $\tau > 0$ we can give the following proof of Theorem 1.3 to establish the iteration formulae of the Maslov-type index theory for any paths in $\mathrm{Sp}(2n)$.

Proof of Theorem 1.3. By above Theorem 1.2, Lemmas 2.3 and 2.4, and our studies in the section 2 and 3, summing these results up we obtain (1.7) and (1.8). ∎

4. Other iteration properties

Based on the iteration formulae Theorem 1.3, we briefly indicate how other iteration properties of the Maslov-type index theory for any paths in $\mathrm{Sp}(2n)$ can be derived from Theorem 1.3.

Corollary 4.1. *Using notations of Theorem 1.3, for any $\gamma \in \mathcal{P}_\tau(2n)$ there hold*

$$\hat{i}_\tau(\gamma) \equiv \lim_{m\to+\infty} \frac{i_{m\tau}(\gamma^m)}{m} = i_\tau(\gamma) + p_- + p_0 - r + \sum_{j=1}^{r} \frac{\theta_j}{\pi} \in \mathbf{R}, \tag{4.1}$$

$$\lim_{m\to+\infty} \frac{\nu_{m\tau}(\gamma^m)}{m} = 0. \tag{4.2}$$

Here $\hat{i}_\tau(\gamma)$ is the Maslov-tye mean index of γ per τ firstly introduced in [Lo9].

Note that by definition, there holds

$$\hat{i}_{m\tau}(\gamma^m) = m\hat{i}_\tau(\gamma). \tag{4.3}$$

As applications of Theorem 1.3, next we give different proofs to some results in [LL1], [LL2], and [DL].

Corollary 4.2. (Theorem of [LL1], cf. also 1° of Theorem 1.1 of [LL2]) *For any $\tau > 0$, $\gamma \in \mathcal{P}_\tau(2n)$, and $m \in \mathbf{N}$, there holds*

$$m\hat{i}_\tau(\gamma) - n \le i_{m\tau}(\gamma^m) \le m\hat{i}_\tau(\gamma) + n - \nu_{m\tau}(\gamma^m). \tag{4.4}$$

Proof. Using notations in Theorem 1.3, by (4.1) we obtain

$$\nu_\tau(\gamma) - p_- - p_0 - n \le \sum_{j=1}^{r} \theta_j/\pi - r \le n - p_- - p_0.$$

This implies

$$i_\tau(\gamma) + \nu_\tau(\gamma) - n \le \hat{i}_\tau(\gamma) \le i_\tau(\gamma) + n. \tag{4.5}$$

Then by (4.3) and (4.5), we obtain (4.4). ∎

Corollary 4.3. (cf. 1° of Theorem 1.2 of [LL2]) *For any $\tau > 0$, $\gamma \in \mathcal{P}_\tau(2n)$, and $m \in \mathbf{N}$, there holds*

$$\begin{aligned} m(i_\tau(\gamma) + \nu_\tau(\gamma) - n) + n - \nu_\tau(\gamma) &\le i_{m\tau}(\gamma^m) \\ &\le m(i_\tau(\gamma) + n) - n - (\nu_{m\tau}(\gamma^m) - \nu_\tau(\gamma)). \end{aligned} \tag{4.6}$$

Proof. It suffices to note that the iteration formulae established for the five basic cases in the sections 2 and 3 make (4.6) hold respectively. Then by Theorem 1.2 and the symplectic additivity of the Maslov-type index theory given by Theorem 1.4 of [Lo8], we obtain (4.6) for general symplectic paths. ∎

Next we derive certain useful iteration equalities and inequalities from Theorem 1.3. For $m \in \mathbf{N}$ we define

$$\mathrm{Sp}(2n)_m^* = \{M \in \mathrm{Sp}(2n) \,|\, \det(M^m - I) \neq 0\}, \qquad \mathrm{Sp}(2n)_m^0 = \mathrm{Sp}(2n)\backslash \mathrm{Sp}(2n)_m^*.$$

Note that $\mathrm{Sp}(2n)_m^*$ is an open subset of $\mathrm{Sp}(2n)$.

Remark 4.5. Note that the iteration equality Theorem 4.1 of [DL] for non-degenerate symplectic paths, and iteration inequality Theorem 8.3 of [DL] for degenerate symplectic paths can also be derived from our Theorem 1.3 similarly. The details are left to the readers.

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